Lesson 12. Dynamic Programming – Review

- Recall from Lessons 5-11:
  - A **dynamic program** models situations where decisions are made in a sequential process in order to optimize some objective
  - **Stages** $t = 1, 2, \dots, T$
    - stage $T \leftrightarrow$ end of decision process
  - **States** $n = 0, 1, \dots, N \leftarrow$ possible conditions of the system at each stage
  - Two representations: **shortest/longest path** and **recursive**

Shortest/longest path		Recursive
node t_n	$\leftrightarrow$	state n at stage t
edge $(t_n, (t+1)_m)$	$\leftrightarrow$	allowable decision x_t in state n at stage t that results in being in state m at stage $t+1$
length of edge $(t_n, (t+1)_m)$	$\leftrightarrow$	cost/reward of decision x_t in state n at stage t that results in being in state m at stage $t+1$
length of shortest/longest path from node t_n to end node	$\leftrightarrow$	cost/reward-to-go function $f_t(n)$
length of edges (T_n, end)	$\leftrightarrow$	boundary conditions $f_T(n)$
shortest or longest path	$\leftrightarrow$	recursion is min or max: $f_t(n) = \min_{x_t \text{ allowable}} \text{ or } \max \left\{ \begin{pmatrix} \text{cost/reward of} \\ \text{decision } x_t \end{pmatrix} + f_{t+1}\begin{pmatrix} \text{new state} \\ \text{resulting} \\ \text{from } x_t \end{pmatrix} \right\}$
source node 1_n	$\leftrightarrow$	desired cost-to-go function value $f_1(n)$

Example 1. Simplexville Oil needs to build capacity to refine 1,000 barrels of oil and 2,000 barrels of gasoline per day. Simplexville can build a refinery at 2 locations. The cost of building a refinery is as follows:

Oil capacity per day	Gas capacity per day	Building cost ($ millions)
0	0	0
1000	0	5
0	1000	7
1000	1000	14

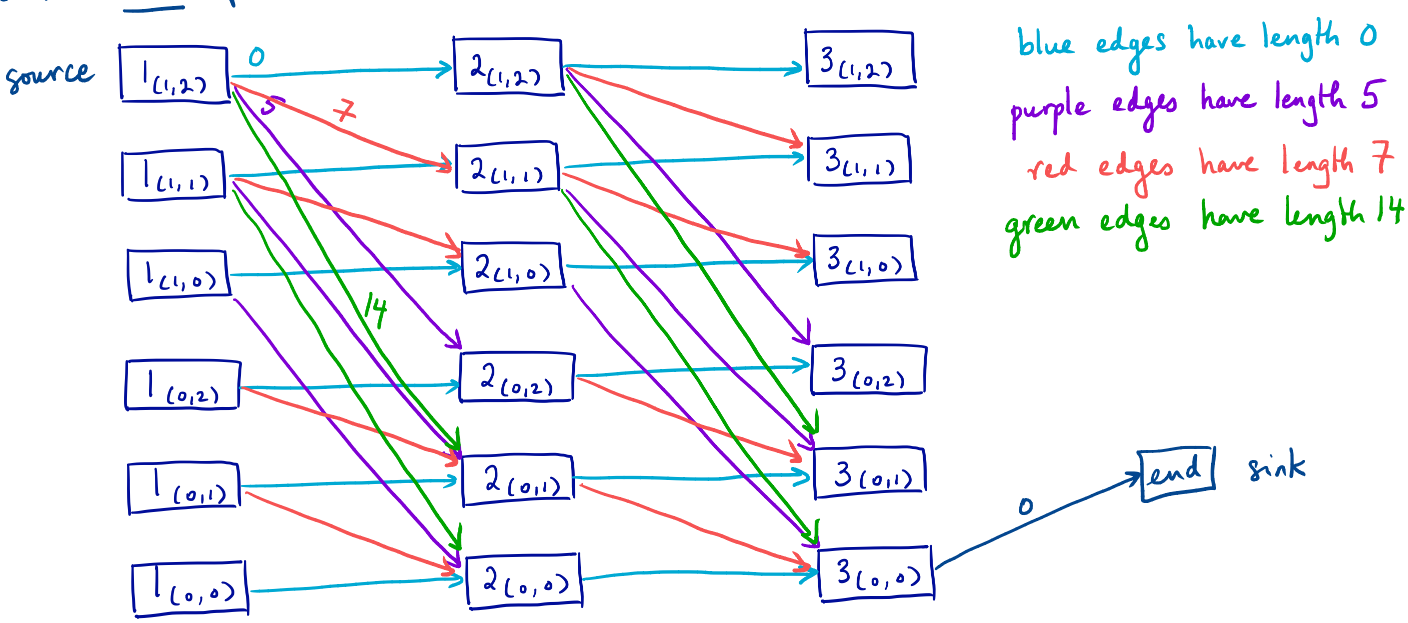
The problem is to determine how much capacity should be built at each location in order to minimize the total building cost. To make things a little simpler, assume that the capacity requirements must be met exactly.

a. Formulate this problem as a dynamic program by giving its shortest path representation.
b. Formulate this problem as a dynamic program by giving its recursive representation. Solve the dynamic program.

Stage t represent deciding to build at location t $(t = 1, 2)$ or the end of the decision-making process $(t = 3)$

State (n_1, n_2) represents having n_1 oil capacity and n_2 gas capacity still needed to be built $(n_1 = 0, 1;\ n_2 = 0, 1, 2)$

Find the <u>shortest</u> path:



Recursive representation

- Stage t represents deciding to build at location t $(t=1,2)$ or the end of the decision-making process $(t=3)$

- State (n_1, n_2) represents having n_1 oil capacity and n_2 gas capacity still needed to be built $(n_1 = 0, 1 \,;\, n_2 = 0, 1, 2)$

- Allowable decisions x_t at stage t and state (n_1, n_2):

  t=1,2: $x_t = (x_{t1}, x_{t2})$ = build x_{t1} oil capacity and x_{t2} gas capacity at location t

  x_t must satisfy:

  $$\begin{aligned} &x_{t1} \in \{0,1\} \\ &x_{t2} \in \{0,1\} \\ &\left.\begin{aligned} x_{t1} &\le n_1 \\ x_{t2} &\le n_2 \end{aligned}\right\} \text{ can't overbuild capacity.} \end{aligned}$$

  t=3: no decisions

- Cost of x_t at stage t and state (n_1, n_2):

$$c(x_{t1}, x_{t2}) = \begin{cases} 0 & \text{if } (x_{t1}, x_{t2}) = (0,0) \\ 5 & \text{if } (x_{t1}, x_{t2}) = (1,0) \\ 7 & \text{if } (x_{t1}, x_{t2}) = (0,1) \\ 14 & \text{if } (x_{t1}, x_{t2}) = (1,1) \end{cases} \qquad \text{for } t=1,2,\; n_1 = 0,1,\; n_2 = 0,1,2$$

- Cost-to-go function:

  $f_t(n_1, n_2)$ = minimum cost to build n_1 oil and n_2 gas capacity with locations $t, t+1, \ldots$ available — for $t=1,2,3$; $n_1 = 0,1$; $n_2 = 0,1,2$

- Boundary conditions:

$$f_3(n_1, n_2) = \begin{cases} 0 & \text{if } (n_1, n_2) = (0,0) \\ +\infty & \text{o/w} \end{cases} \qquad \text{for } n_1 = 0,1,\; n_2 = 0,1,2$$

- Recursion:

$$f_t(n_1, n_2) = \min_{\substack{x_{t1} \in \{0,1\} \\ x_{t2} \in \{0,1\} \\ x_{t1} \le n_1,\; x_{t2} \le n_2}} \left\{ c(x_{t1}, x_{t2}) + f_{t+1}(n_1 - x_{t1}, n_2 - x_{t2}) \right\} \qquad \text{for } t=1,2,\; n_1 = 0,1,\; n_2 = 0,1,2$$

- Desired cost-to-go value: $f_1(1,2)$

Solving backwards:

$$\boxed{f_t(n_1,n_2) = \min_{\substack{x_{t1}\in\{0,1\} \\ x_{t2}\in\{0,1\} \\ x_{t1}\le n_1,\ x_{t2}\le n_2}} \left\{ c(x_{t1},x_{t2}) + f_{t+1}(n_1 - x_{t1}, n_2 - x_{t2}) \right\}}$$

Stage 3: (boundary conditions)

$$f_3(n_1,n_2) = \begin{cases} 0 & \text{if } (n_1,n_2)=(0,0) \\ +\infty & \text{o/w} \end{cases} \qquad \text{for } \begin{array}{l} n_1 = 0,1 \\ n_2 = 0,1,2 \end{array}$$

Stage 2:

$$f_2(1,2) = \min\left\{ \underset{(0,0)}{\overset{0+\infty=+\infty}{c(0,0)+f_3(1,2)}},\ \underset{(1,0)}{\overset{5+\infty=+\infty}{c(1,0)+f_3(0,2)}},\ \underset{(0,1)}{\overset{7+\infty=+\infty}{c(0,1)+f_3(1,1)}},\ \underset{(1,1)}{\overset{14+\infty=+\infty}{c(1,1)+f_3(0,1)}} \right\} = +\infty$$

$$f_2(1,1) = \min\left\{ \underset{(0,0)}{\overset{0+\infty=+\infty}{c(0,0)+f_3(1,1)}},\ \underset{(1,0)}{\overset{5+\infty=+\infty}{c(1,0)+f_3(0,1)}},\ \underset{(0,1)}{\overset{7+\infty=+\infty}{c(0,1)+f_3(1,0)}},\ \underset{(1,1)}{\overset{14+0=14}{c(1,1)+f_3(0,0)}} \right\} = 14$$

$$f_2(1,0) = \min\left\{ \underset{(0,0)}{\overset{0+\infty=+\infty}{c(0,0)+f_3(1,0)}},\ \underset{(1,0)}{\overset{5+0=5}{c(1,0)+f_3(0,0)}} \right\} = 5$$

$$f_2(0,2) = \min\left\{ \underset{(0,0)}{\overset{0+\infty=+\infty}{c(0,0)+f_3(0,2)}},\ \underset{(0,1)}{\overset{7+\infty=+\infty}{c(0,1)+f_3(0,1)}} \right\} = +\infty$$

$$f_2(0,1) = \min\left\{ \underset{(0,0)}{\overset{0+\infty=+\infty}{c(0,0)+f_3(0,1)}},\ \underset{(0,1)}{\overset{7+0=7}{c(0,1)+f_3(0,0)}} \right\} = 7$$

$$f_2(0,0) = \min\left\{ \underset{(0,0)}{\overset{0+0=0}{c(0,0)+f_3(0,0)}} \right\} = 0$$

Stage 1:

$$f_1(1,2) = \min\left\{ \underset{(0,0)}{\overset{0+\infty=+\infty}{c(0,0)+f_2(1,2)}},\ \underset{(1,0)}{\overset{5+\infty=+\infty}{c(1,0)+f_2(0,2)}},\ \underset{(0,1)}{\overset{7+14=21}{c(0,1)+f_2(1,1)}},\ \underset{(1,1)}{\overset{14+7=21}{c(1,1)+f_2(0,1)}} \right\} = 21$$

$\Rightarrow$ Optimal solution: $x_1 = (1,1)$ $\quad x_2 = (0,1)$ $\longleftarrow$ Build at location 1: 1000 oil, 1000 gas capacity

Build at location 2: 1000 gas capacity

Optimal value: 21 $\longleftarrow$ minimum total cost = \$21M

Note: $x_1 = (0,1)$, $x_2 = (1,1)$ is also an optimal solution